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Numerical Analysis of Eigenvalue Solution of Disk Resonator

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***Abstract*—A formulation is proposed to calculate the frequencies of the eigenmodes for a resonator with a thin conductor disk placed in the median plane between two infinite parallel conductor plates. The numerical analysis is carried out for the E- and EH-modes, and these eigenvalues are calculated as the function of the ratio of the disk radius to the distance between the disk and one of the infinite conductor plates. It is shown that at a ratio greater than a certain value the exact eigenvalue is smaller than the one predicted by applying the conventional method for two-dimensional bifurcation of rectangular waveguide, but the latter becomes closer to the exact one with increasing ratio. The availability of our exact eigenvalues is demonstrated in determining experimentally the dielectric constant of Teflon plate specimen by applying those values. Then the constancy of the measured dielectric constant is confirmed irrespective of the modes and the ratios.**

I. INTRODUCTION

In the recent experiments by Kobayashi *et al.* [1], [2] it is shown that the high-precision measurement of the resonance frequencies of the modes excited in the resonator containing dielectric plates between two sufficiently large parallel plates and a thin conductor disk placed in the median plane (see Fig. 1.) provides a possible method to determine accurately the dielectric constant. Present analysis is motivated by the facts that this accuracy depends upon the correctness of the relation between the measured frequency and the dielectric constant, as well as that the disk resonator plays an important role particularly in the planar circuits [3].

We consider the resonant cavity immersed with a dielectric substance of permittivity ϵ and permeability μ, assumed later to be vacuum value, as shown in Fig. 1. For simplicity the thickness of the disk is neglected, and the conductivity of the plates and the disk is assumed to be infinite. The axially symmetric property of such device is conveniently described in the cylindrical coordinates whose origin is at the center of the disk and z axis perpendicular to the plates and the disk.

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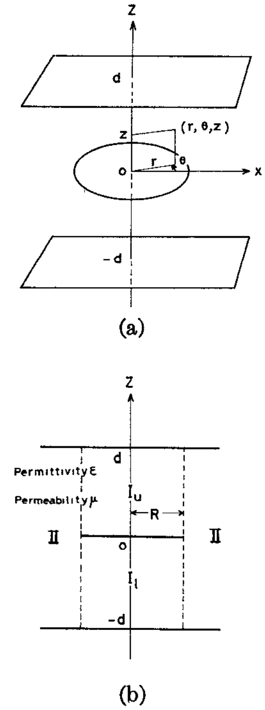

Fig. 1. Disk resonator. (a) General view. (b) Side view.

It is well known that as the ratio $s = R/d$ for the disk radius R and the distance between the disk and one of the parallel plates become large, the eigenvalue of the E_{nml}-mode defined by $x = kR$ with $k = \omega(\epsilon\mu)^{1/2}$ approaches the mth root of the equation

$$J_n'(x_{nm}) = 0 \tag{1}$$

which is derived simply by requiring the boundary condition for an open circuit at disk edge. When s is not too small, the dependence on s must be taken into account by applying the conventional method for the two-dimensional bifurcation of a rectangular waveguide [4], which we refer to simply as the Marcuvitz method hereafter. Then the eigenvalue x_{nm} is approximated by the root of the equation given by

$$J_n'(x_{nm}') = 0 \tag{2}$$

with

$$x_{nm}' = \left(1 + \frac{2 \ln 2}{\pi s}\right) x_{nm} + S_1\left(\frac{2x_{nm}}{\pi s}; 0,0\right) - 2S_1\left(\frac{x_{nm}}{\pi s}; 0,0\right) \tag{3}$$

where the function S_1 is defined by

$$S_1(z; 0,0) = \sum_{n=1}^{\infty} [\sin^{-1}(z/n) - z/n]. \tag{4}$$

Obviously (2) reduces to (1) as s tends to infinity. When we neglect the second and third terms in (4), the solution of (2) corresponds to the one of (1) for the disk radius effectively enlarged by $2 \ln 2d/\pi$ [5], [6]. We refer to this approximation as the approximated Marcuvitz method.

II. EIGENVALUE FORMULATION

For a loss-free medium all the field components can be expressed in terms of the z components of Hertz vectors Π_e for electric mode and Π_m for magnetic mode satisfying the same Helmholtz equations

$$(\Delta + k^2)\Pi_e = 0 \quad \text{and} \quad (\Delta + k^2)\Pi_m = 0. \tag{5}$$

The most general solutions satisfying the appropriate boundary conditions are provided by

$$\Pi_e^{\text{in}} = \pm \sum_{\mu=1}^{\infty} A_\mu J_n(k_{2(\mu-1)}r) \cos\frac{(\mu-1)\pi z}{d} \cos n\theta \quad (6)$$

$$\Pi_m^{\text{in}} = \pm \sum_{\mu=1}^{\infty} C_\mu J_n(k_{2(\mu-1)}r) \sin\frac{(\mu-1)\pi z}{d} \sin n\theta \quad (7)$$

for the interior region, where the upper and lower signs are for the regions $I_u(0 \le r \le R,\ 0 < z \le d)$ and $I_l(0 \le r \le R, -d \le z < 0)$, respectively; and

$$\Pi_e^{\text{ex}} = \sum_{\nu=1}^{\infty} B_\nu H_n^{(1)}(k_{2\nu-1}r) \sin\frac{(2\nu-1)\pi z}{2d} \cos n\theta, \quad (8)$$

$$\Pi_m^{\text{ex}} = \sum_{\nu=1}^{\infty} D_\nu H_n^{(1)}(k_{2\nu-1}r) \cos\frac{(2\nu-1)\pi z}{2d} \sin n\theta \quad (9)$$

for the exterior region $\mathrm{II}(r > R, -d \le z \le d)$. In the above, the constants A, B, C, and D are, so far, arbitrary, and n is a nonnegative integer. We have introduced the quantity

$$k_\kappa^2 = k^2 - (\kappa\pi/2d)^2 \quad (10)$$

for $\kappa = 0,1,2,\cdots$.

Then all the field components for both the interior and exterior regions are calculated by properly differentiating Π^{in} and Π^{ex} [7]. It is easily seen that those field components satisfy all the necessary boundary conditions required on the conductor surfaces, namely $E_r^{\text{in}} = E_\theta^{\text{in}} = 0$ at $z = 0$ and $\pm d$ for $0 \le r \le R$, and $E_r^{\text{ex}} = E_\theta^{\text{ex}} = 0$ at $z = \pm d$ for $r > R$. Those solutions are the most general in the extent that the interior solution is finite at $r = 0$, and the exterior solution damps fast enough not to allow the energy leakage, because the first kind Hankel functions of positive imaginary arguments are exponentially decreasing at large value of r. Both the interior and exterior solutions depend on θ in common, but are the different Fourier series with respect to z. The continuities of all the field components at the boundary defined by $r = R$ and $-d \le z \le d$ are guaranteed if the components E_z, H_r, H_θ, and H_z are continuous there.

A. The Electric Mode

The case of $n = 0$ is exceptional, since all the magnetic contributions vanish, namely the terms with the coefficients C and D do not exist in the expressions for the field components, so that we have only the E-type mode. Then the components H_r and H_z vanish everywhere, and for the remaining components we require $E_z^{\text{in}} = E_z^{\text{ex}}$ and $H_\theta^{\text{in}} = H_\theta^{\text{ex}}$ at $r = R$ for any z and θ. Each of these equations can be regarded as the Fourier series expansion of the odd function of z in the left-hand side.

Eliminating the expansion coefficients from these equations we immediately arrive at the following equation for the E_{0ml}-mode eigenvalues:

$$\det M = 0 \quad (11)$$

where M is the matrix with infinite number of rows and columns, whose element will be given in what follows.

Here we introduce the notations

$$x_0 = x = kR \quad \text{and} \quad x_\kappa = [(\kappa\pi s/2)^2 - x^2]^{1/2} \quad (12)$$

for $\kappa = 1,2,\cdots$. According to the range to which the value of x belongs, namely $(2l-1)\pi s/2 < x < (2l+1)\pi s/2$ for $l = 0, 1, 2,\cdots$, we replace the J-functions of imaginary arguments by corresponding I-functions to make all the matrix elements real. Since the asymptotic behaviors of the $H^{(1)}$-functions of real arguments are inappropriate for resonant solution, those terms do not exist, namely $B_1 = B_3 = \cdots = B_{2l-1} = 0$. Then the arguments $x_\kappa = j[x^2 + (\kappa\pi s/2)^2]^{1/2}$ for $\kappa = 1,2,\cdots, 2l-1$ are purely imaginary. Therefore the matrix elements are given in the following alternate forms according to the different ranges for the value x:

$$M_{\rho\sigma} = \frac{1}{x_{2\rho+2l-1}^2 + |x_{2(\sigma-1)}|^2}\left[\frac{1}{|x_{2(\sigma-1)}|}\frac{J_0'(|x_{2(\sigma-1)}|)}{J_0(|x_{2(\sigma-1)}|)} + \frac{1}{x_{2\rho+2l-1}}\frac{K_0'(x_{2\rho+2l-1})}{K_0(x_{2\rho+2l-1})}\right] \quad (13)$$

for $\rho = 1,2,\cdots$ and $\sigma = 1,2,\cdots,p$; and

$$M_{\rho\sigma} = \frac{1}{x_{2\rho+2l-1}^2 - x_{2(\sigma-1)}^2}\left[\frac{1}{x_{2(\sigma-1)}}\frac{I_n'(x_{2(\sigma-1)})}{I_n(x_{2(\sigma-1)})} - \frac{1}{x_{2\rho+2l-1}}\frac{K_0'(x_{2\rho+2l-1})}{K_0(x_{2\rho+2l-1})}\right] \quad (14)$$

for $\rho = 1,2,\cdots$ and $\sigma = p+1,\ p+2,\cdots$, where an integer p is defined by

$$p = \begin{cases} l, & \text{if } (2l-1)\pi s/2 < x < l\pi s \\ l+1, & \text{if } l\pi s < x < (2l+1)\pi s/2. \end{cases} \quad (15)$$

B. The Hybrid Mode

When n is positive integer, the electric and magnetic modes are hybrid ones. Since they possess nonvanishing E_z and H_z components, we call them EH-modes. Then, the four requirements $H_r^{\text{in}} = H_r^{\text{ex}}$, $H_\theta^{\text{in}} = H_\theta^{\text{ex}}$, $E_z^{\text{in}} = E_z^{\text{ex}}$, and $H_z^{\text{in}} = H_z^{\text{ex}}$ at $r = R$ for any z and θ provide, respectively, the four independent equations. Eliminating all the coefficients A, B, C, and D from those, we finally obtain the eigenvalue equation analogous to (11) for the electric mode. Thus, the eigenvalue equation is

$$\det N = 0 \quad (16)$$

whose matrix elements are given by

$$N_{2\rho-1,2\sigma-1} = \frac{x_{2\rho+2l-1}^2 |x_{2(\sigma-1)}|^2}{x_{2\rho+2l-1}^2 + |x_{2(\sigma-1)}|^2}\left[\frac{1}{|x_{2(\sigma-1)}|}\frac{J_n'(|x_{2(\sigma-1)}|)}{J_n(|x_{2(\sigma-1)}|)} + \frac{1}{x_{2\rho+2l-1}}\frac{K_n'(x_{2\rho+2l-1})}{K_n(x_{2\rho+2l-1})}\right] \quad (17)$$

for $\rho = 1, 2,\cdots$ and $\sigma = 1, 2,\cdots,p$; and

$$N_{2\rho-1,2\sigma-1} = \frac{x_{2\rho+2l-1}^2 x_{2(\sigma-1)}^2}{x_{2\rho+2l-1}^2 - x_{2(\sigma-1)}^2}\left[\frac{1}{x_{2(\sigma-1)}}\frac{I_n'(x_{2(\sigma-1)})}{I_n(x_{2(\sigma-1)})} - \frac{1}{x_{2\rho+2l-1}}\frac{K_n'(x_{2\rho+2l-1})}{K_n(x_{2\rho+2l-1})}\right] \quad (18)$$

for $\rho = 1,2,\cdots$ and $\sigma = p+1,\ p+2,\cdots$, where an integer p is the same as defined by (15). Furthermore,

$$N_{2\rho-1,2\sigma} = N_{2\rho,2\sigma-1} = n \quad (19)$$

and

$$N_{2\rho,2\sigma} = N_{2\rho-1,2\sigma+1} \quad (20)$$

for $\rho,\sigma = 1,2,\cdots$.

III. NUMERICAL ANALYSIS

A. The E_{0m0}-Mode

To solve (11) we approximate the infinite-dimensional determinant with the finite one composed of the finite square section in the upper-left corner of the original matrix M. Though we expect the convergence of the series assumed for the solutions, the propriety of this expectation must be ascertained by investigating the dependence of the solution on the dimensionality of the determinant taken in the actual numerical analysis. For this purpose we calculate the eigenvalues for several dimensionalities, and compare the results to infer the inherent calculational errors. The calculated

E_{010}-mode eigenvalues at various ratios are summarized in Table I. The dimensionalities of the approximate determinant are indicated in the title lines of the respective columns for the solutions of (11). The solutions in the Marcuvitz method (2) and the approximated Marcuvitz method are listed in the sixth and seventh columns, respectively. For this mode there exists the cutoff value $x = 2.4048$ at $s = 1.5308$. The solutions are compared with the value 3.8317 in the open circuit approximation (1).

It is observed that the exact solution increases monotonically with increasing dimensionality, and the convergence of the solution is better at large values of s. It is remarked that the solution in the Marcuvitz method is larger than the exact value, but becomes closer to the latter asymptotically as s increases. However, the Marcuvitz method predicts a value smaller than the exact one at small s. This crossover point is $s = 4.2$, and the Marcuvitz method provides unexpectedly precise values in this vicinity.

B. The EH_{nm0}-Mode

The eigenvalues for the hybrid modes are calculated by solving (16), and the results for the EH_{110}-, EH_{210}-, and EH_{310}-modes are summarized in Tables II, III, and IV, respectively. Since we could confirm the reliability of our calculational method in the previous case, here we cite only the results derived from 80-dimensional determinants for those cases. The solutions in the Marcuvitz method and the approximated Marcuvitz method are also listed in the third and fourth columns, respectively.

It is found that there exists a cutoff point also for each hybrid mode, while it is ignored in the approximated solution from the open-circuit condition. In Tables II, III, and IV the dashes represent the nonexistence of the eigenvalue because of the cutoff. Comparing

TABLE I
EIGENVALUES OF E_{010}-MODE

Ratio s = R/d	Exact Solution $x_{010}(s)$				Marcuvitz Method	Approximated Marcuvitz Method
	20 X 20	30 X 30	40 X 40	60 X 60		
2.0	2.9404	2.9427	2.9439	2.9451	2.9307	3.1391
3.0	3.2777	3.2801	3.2813	3.2825	3.2811	3.3404
5.0	3.5030	3.5047	3.5055	3.5064	3.5073	3.5210
10.0	3.6652	3.6661	3.6666	3.6670	3.6679	3.6698
15.0	3.7197	3.7204	3.7207	3.7210	3.7216	3.7222
20.0	3.7473	3.7478	3.7480	3.7483	3.7487	3.7490
25.0	3.7639	3.7643	3.7645	3.7647	3.7651	3.7652
30.0	3.7751	3.7754	3.7756		3.7761	3.7762
40.0	3.7891	3.7894	3.7895		3.7899	3.7899
50.0	3.7976	3.7978	3.7979		3.7982	3.7982

x_{010} = 3.8317 from open circuit condition

TABLE II
EIGENVALUES OF EH_{110}-MODE

Ratio s = R/d	Exact Solution $x_{110}(s)$ 80 X 80	Marcuvitz Method	Approximated Marcuvitz Method
0.5	———	0.77238	0.97804
0.8	1.1223	1.0810	1.1867
1.0	1.2404	1.2080	1.2775
2.0	1.4986	1.4935	1.5084
3.0	1.5998	1.5997	1.6051
5.0	1.6891	1.6905	1.6919
10.0	1.7621	1.7632	1.7634
15.0	1.7877	1.7885	1.7886
20.0	1.8008	1.8014	1.8015
25.0	1.8087	1.8092	1.8093
30.0	1.8141	1.8145	1.8145
40.0		1.8211	1.8211
50.0		1.8251	1.8251

x_{110} = 1.8412 from open circuit condition

TABLE III
EIGENVALUES OF EH_{210}-MODE

Ratio s = R/d	Exact Solution $x_{210}(s)$ 80 X 80	Marcuvitz Method	Approximated Marcuvitz Method
1.7	———	2.2952	2.4245
2.0	2.4565	2.4196	2.5021
3.0	2.6439	2.6354	2.6626
5.0	2.7987	2.7999	2.8065
10.0	2.9220	2.9242	2.9251
15.0	2.9649	2.9667	2.9669
20.0	2.9868	2.9882	2.9883
25.0	3.0001	3.0012	3.0012
30.0	3.0090	3.0099	3.0099
40.0		3.0209	3.0209
50.0		3.0275	3.0275

x_{210} = 3.0542 from open circuit condition

TABLE IV
EIGENVALUES OF EH_{310}-MODE

Ratio s = R/d	Exact Solution $x_{310}(s)$ 80 X 80	Marcuvitz Method	Approximated Marcuvitz Method
2.0	———	3.1049	3.4418
2.3	3.4300	3.3275	3.5249
3.0	3.6182	3.5794	3.6625
5.0	3.8449	3.8421	3.8605
10.0	4.0180	4.0211	4.0237
15.0	4.0776	4.0804	4.0811
20.0	4.1079	4.1102	4.1105
25.0	4.1263	4.1281	4.1283
30.0	4.1386	4.1402	4.1403
40.0		4.1553	4.1554
50.0		4.1553	4.1554

x_{310} = 4.2012 from open circuit condition

the predicted values by the Marcuvitz method with our exact ones we find that the former method provides an excellent approximation for the eigenvalues at large s. However, it gives too small values at small s, as well as fails to predict the correct cutoff point.

The required computation time for the above calculation was about 100 s per each eigenvalue from 80-dimensional determinant on a HITAC 8800/8700 computer system at the University of Tokyo.

IV. MEASUREMENT OF DIELECTRIC CONSTANT

We attempt to measure the dielectric constant of Teflon plates of four different thicknesses using the lowest modes EH_{110}, EH_{210}, EH_{010}, and EH_{310}. A disk resonator is constructed with a copper disk of thickness 0.05 mm placed in a form of balanced stripline between two Teflon substrates of same thickness. Light coupling to the disk resonator is achieved by E-field probes composed of the stripline of open circuit, and the transmitted power is detected as resonance occurs. The results are shown in Fig. 2.

Making use of the following formula:

$$\epsilon_r = \left[\frac{\lambda_0}{2\pi R} x_{nm0}(s)\right]^2 \qquad (21)$$

where $\lambda_0 = 2\pi c/\omega$ and c is light velocity we determined the dielectric constant ϵ_r from each measured value of resonant wavelength λ_0 shown in Fig. 2.

The average of all the measurements gave $\epsilon_r = 2.043$ and the

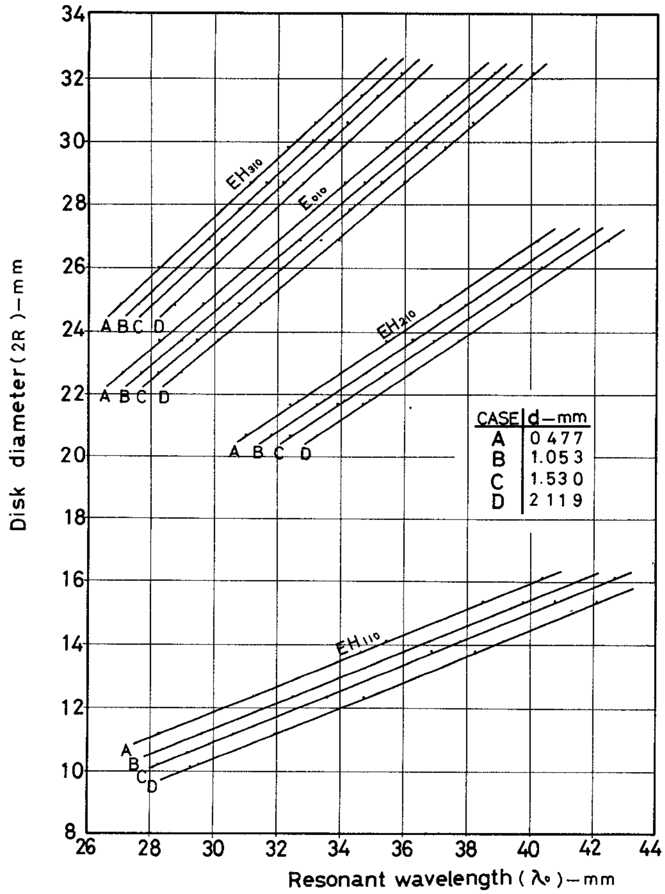

Fig. 2. The measured resonant wavelength for some lowest modes versus disk diameter in the four cases of teflon thickness (d).

probable error 0.2 percent. We could not find any appreciable discrepancies among experimental values measured at various ratios between 5 and 30 for the different modes.

V. CONCLUSION

We have carried out the numerical analysis only for a few lowest E- and EH-modes, which can be easily excited by simple means [1], [2], [8] and may be useful for the practical usages. However, we remark that the HE-mode also obeys (16), and its eigenvalue is obtainable in a similar way. Thus our method of series expansion can be utilized for any resonance mode of our disk resonator.

The applicabilities of the Marcuvitz method and the approximated Marcuvitz method are limited to the region of large ratios. In particular, the latter contains considerable error at small ratios. The difficulty of finding some approximation method at small ratios is implied by slow convergence to the exact value of eigenvalue according as increasing dimensionality of approximate determinant, namely by the significant admixture of the higher harmonics.

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A New Class of Nonreciprocal Components Using Slot Line

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***Abstract*—The authors present an experimental and theoretical study of edge modes in a ferrite-loaded slot line. Nonreciprocal properties are obtained over a broad frequency band. Added microstrip lines provide suitable transitions.**

A theory based on magnetic boundary conditions shows good agreement with the experimental results and allows a comparison with stripline devices described by Hines. In particular, the characteristics of the slot-line isolators are satisfactorily explained by this theory.

I. INTRODUCTION

It is well known [1], [2] that the dominant mode in a slot line exhibits regions where the magnetic field is elliptically polarized both in the substrate and in the air outside the substrate. It is possible to utilize this property to the design of nonreciprocal microwave components such as isolators using a resonance effect. However, such a device does not offer a broad frequency band of operation.

In this short paper we will demonstrate that it is possible to obtain nonreciprocal behavior due to an integral field-displacement effect over a broad frequency band with a device using a slot line on a ferromagnetic substrate [3].

II. EXPERIMENTAL STUDY OF THE DEVICE

Our initial experiments with various slot-line geometries led to the selection of the suitable structure shown in Fig. 1. Using such a device, we have performed experiments using liquid crystals in order to demonstrate the field-displacement effect. Liquid crystals are deposited on the two sides of the device, with the absorbent removed, and they act as a transducer permitting visualization of the electromagnetic energy [4]. If we designate an input and an output, it appears that for a given applied dc field, the direct energy is traveling along one side while the reverse energy is traveling along the other side. Consequently, we can hypothesize an energy distribution inside the ferrite, as shown in Fig. 1(b). The alumina substrate in Fig. 1(a) is especially helpful in reducing the insertion losses of the device.

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